TOPOLOGY-III
M.MATH-II
FINAL EXAM

Total 50 Marks

(1) Show that the cohomology ring $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[t]/(t^{n+1})$. (7 marks)

(2) Show that there is no continuous map $f : \mathbb{RP}^{n+1} \to \mathbb{RP}^n$ such that the induced map $f_* : H_1(\mathbb{RP}^{n+1};\mathbb{Z}_2) \to H_1(\mathbb{RP}^n;\mathbb{Z}_2)$ is an isomorphism.
(Hint: Use the the cohomology ring structure of $\mathbb{RP}^n$ described in problem (1).)
(7 marks)

(3) Prove that a manifold M is orientable if there is no index 2 subgroup of $\pi_1(M)$. Give an example where the converse is not true. (5 marks)

(4) Compute the homotopy groups $\pi_n(S^1)$ for $n > 1$. (5 marks)

(5) Let M be a orientable closed manifold of dimension n. Show that the group $H_{n-1}(M,\mathbb{Z})$ is torsion free.
(Hint: Homology groups of compact manifolds are finitely generated.) (7 marks)

(6) Prove that if X is a CW-complex with $\pi_n(X) = 0$ for all n then X is contractible.
(5 marks)

(7) Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M;\mathbb{Z})$ torsion free, then $H_k(M;\mathbb{Z})$ is also torsion free. (7 marks)

(8) Compute the cohomology ring $H^*(S^n \times S^m;\mathbb{Z})$. (7 marks)

Hint for problem (1) and (8): The cup product paring $H^k(M;R)\times H^{n-k}(M;R) \to H^n(M;R)$ is non singular for closed R-orientable manifolds when R is a field, or when $R = \mathbb{Z}$ and torsion in $H^*(M,\mathbb{Z})$ is factored out.